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On Semi-groups Arising from Quantum Groups

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Let G_q be a quantum group. In this paper, we introduce and completely characterize a semi-group and its Borel sub-semi-groups consisting of algebra homomorphisms from the coordinate algebra $k[G_q]$ of G_q. Also, we present invariant elements for the semi-group, which are closely connected to Hopf ideals of $k[G_q]$. Finally, based on the theory of structures of the semi-group, we prove the existence of a Frobenius morphism over G_q whenever the parameter q is a root of unity. © 1999 Academic Press

INTRODUCTION

The approach to quantum groups includes the study of coordinate algebras related to Yang–Baxter operators. This was carried out only for the case of type A by Parshall and Wang [3], although all Yang–Baxter operators of classical types had been found a long time before. The difficulty lies in the complexity of coordinate algebras of quantum groups. To solve this, we develop the theory of semi-groups with generators respecting structures of some closed subgroups of the quantum groups, which resorts to techniques from algebraic groups. In developing the theory, root systems emerge in the form of indices and are closely related to structures of coordinate algebras of quantum groups. As applications of the theory, aside from a success in the proof of an important theorem, i.e., the theorem of density of big cell, the existence of Frobenius morphisms for quantum groups with parameters of roots of unity is also verified.

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Fix a field k in any characteristic. In this paper, we adopt the "naive" point of view of [3], identifying the category QGr_k of quantum groups over k with the dual of the category k-Hopf of finitely generated k-Hopf algebras. The Hopf algebra corresponding to a quantum group G_q, denoted by $k[G_q]$, is called the coordinate algebra of G_q. Moreover, let H and H_1 be two quantum groups. Then H_1 is called as a subgroup of H whenever there is a surjective Hopf algebra homomorphism $k[H] \to k[H_1]$.

Let G_q be a fixed quantum group. We organize the paper as follows. Section 1 presents some concepts, e.g., generic point and root system, etc. Section 2 introduces a fundamental semi-group associated to G_q and two of the Borel sub-semi-groups. Also we develop the basic theory on the semi-group, which includes the connection of some distinguished elements, called invariant elements in the paper, of the semi-group and Hopf ideals of $k[G_q]$. Sections 3, 4, and 5 are devoted to determining the invariant elements of the semi-group and its Borel sub-semi-groups. As a q-analogue of the affine group, a semi-simple quantum group is defined, based on the density of "big cells" in Section 5. Section 6 characterizes entirely structures of the semi-group and the Borel sub-semi-groups. Finally, Section 7 proves that there is a surjective homomorphism from the semi-group associated to G_q to that associated to an algebraic group, which is equivalent to the existence of the Frobenius morphism from G_q to the algebraic group, in case q is a root of unity.

In a sequel, we shall prove that quantum groups of classical types, associated to Yang–Baxter operators, are essentially determined by these semi-groups.

1. THE GENERIC POINTS AND THE ROOT SYSTEM

Let K be a quasi-polynomial algebra over k with generators $\{t, t^{-1}, x\}$ subject to the relations

$$tt^{-1} = t^{-1}t = 1, \qquad xt = qtx.$$

Then, we can make K into two Hopf algebras, denoted by K_1 and K_2, respectively. The associated comultiplications Δ_1 and Δ_2, and antipodes γ_1 and γ_2 are respectively defined as follows. Δ_1 sends t to $t \otimes t$, and x to $t^{-1} \otimes x + x \otimes t$; Δ_2 sends t to $t \otimes t$ and x to $t \otimes x + x \otimes t^{-1}$; γ_1 sends t to t^{-1} and x to $q^{-1}x$, and γ_2 sends t to t^{-1} and x to qx.

DEFINITION 1.1. Let G_q be a quantum group with a maximally diagonal torus T and $X(T)$ the character group of T. If there are a surjective Hopf algebra homomorphism ϕ: $k[G_q] \to K_1$ (resp. ϕ: $k[G_q] \to K_2$) and a Hopf algebra homomorphism ϕ_T: $k[T] \to K$ sending a unique element $\alpha \in X(T)$ to t^2 (resp. t^{-2}) such that the following commutative diagram holds

$$\begin{array}{ccc} k[G_q] & \xrightarrow{\phi} & K_i \\ \downarrow & & \downarrow \\ k[T] & \xrightarrow{\phi_T} & K_i \end{array}$$

for $i = 1$ (resp. $i = 2$), where two vertical maps are canonical, then ϕ is called as a positive (resp. negative) generic-point of G_q associated to α or simply called an α-point in the language of algebraic geometry. Denote by $P(G_q)$, $P^+(G_q)$, and $P^-(G_q)$, respectively, the set of generic-points, positive generic-points, and negative generic-points of G_q, respectively. Usually, an α-point of G_q is written as ϕ_α for $\alpha \in X(T)$.

DEFINITION 1.2. (1) Let G_q be a quantum group with a maximally diagonal torus T; let $R(G_q)$, $R^+(G_q)$, and $R^-(G_q)$ be subsets of $X(T)$ such that

$$P(G_q) = \{\phi_\alpha | \alpha \in R(G_q)\},$$
$$P^+(G_q) = \{\phi_\alpha | \alpha \in R^+(G_q)\}$$
$$P^-(G_q) = \{\phi_\alpha | \alpha \in R^-(G_q)\}.$$

Then $R(G_q) = R^+(G_q) \cup R^-(G_q)$ is clearly a partition; $R(G_q)$, $R^+(G_q)$, and $R^-(G_q)$ are defined as a root system, a positive system, and a negative system of G_q, respectively.

(2) Let $\alpha \in R^+(G_q)$. α is called as a simple root of G_q if there are no $\alpha_1, \alpha_2 \in R^+(G_q)$ such that $\alpha = \alpha_1 + \alpha_2$; denote by $\Pi(G_q)$ the set of simple roots of G_q. A generic point, corresponding to a simple root, is called a simply generic point. Denote by SP the set of simply generic points. Also, put $SP^- = \{\phi_{-\alpha} \in P^-(G_q) | \alpha \in \Pi(G_q)\}$.

(3) For $\alpha \in R^+(G_q)$ and an algebraic group homomorphism f: $k^\times \to T$, define $\langle \alpha, f \rangle$ to be an element in Z such that $\alpha f(m) = m^{\langle \alpha, f \rangle}$ for any $m \in K^\times$ where $k^\times$ is the multiplicative group of units of k. In particular, there is an f, called the coroot of α and denoted by α^v, such that $\langle \alpha, f \rangle = 2$.

In the remainder of this paper, we shall assume that $\Pi(G_q)$ is identified with a simple laced system of simple roots.

2. THE SEMI-GROUP AND THE BOREL SUB-SEMI-GROUPS

Fix a quantum group G_q with a maximally diagonal torus T throughout this section. Let $I(G_q)$ be the set of all ideals of $k[G_q]$ and $F(G_q)$ the set of all unity-preserving homomorphisms of k-algebras from $k[G_q]$. Then there is a map ker: $F(G_q) \to I(G_q)$ with $\ker(f)$ denoting the kernel of f for $f \in F(G_q)$. Define an equivalence relation Γ_q on $F(G_q)$ such that $f \Gamma_q g$ iff $\ker(f) = \ker(g)$ for $f, g \in F(G_q)$ and have then a quotient set $F(G_q)/\Gamma_q$, consisting of equivalence classes of elements, of $F(G_q)$. Also, ker induces a 1-1 map (still written as) ker: $F(G_q)/\Gamma_q \to I(G_q)$.

PROPOSITION 2.1. *There is a multiplication $*$ on $F(G_q)/\Gamma_q$ such that $F(G_q)/\Gamma_q$ becomes a semi-group with the unity ε under it.*

Proof. Let Δ and ε be the comultiplication and the counit of $k[G_q]$, respectively. Define the multiplication $*$ on $F(G_q)/\Gamma_q$ as

$$f_1 * f_2 = (f_1' \otimes f_2')\Delta \qquad \text{for } f_1, f_2 \in F\big(G_q\big)/\Gamma_q,$$

where f_1' and f_2' are representatives for f_1 and f_2, respectively. It is well defined, indeed, if $f_1'' \Gamma_q f_1'$ and $f_2'' \Gamma_q f_2'$. Then it follows that

$$\ker\big((f_1' \otimes f_2')\Delta\big) = \ker\big((f_1'' \otimes f_2'')\Delta\big).$$

As for the associativity of $*$, it follows now from the coassociativity of Δ, while the claim that ε is the unity follows from the fact that ε is the counity of $k[G_q]$, proving the proposition. ∎

Let $BS^+(G_q)$, $BS^-(G_q)$, and $BS(G_q)$ be sub-semi-groups, generated by equivalence classes of elements of $SP(G_q)$, $SP^-(G_q)$, and $SP^-(G_q) \cup SP^+(G_q)$, respectively, of $F(G_q)/\Gamma_q$.

DEFINITION 2.2. $BS(G_q)$, $BS^+(G_q)$, and $BS^-(G_q)$ are called a fundamental semi-group of G_q, a positive Borel sub-semi-group, and a negative Borel sub-semi-group, respectively.

THEOREM 2.3. (1) $\phi_\alpha * \phi_{-\beta} = \phi_{-\beta} * \phi_\alpha$, $\phi_\alpha^2 = \phi_\alpha$, *and* $\phi_{-\alpha}^2 = \phi_{-\alpha}$ *for* $\alpha \neq \beta \in \Pi(G_q)$.

(2) *The multiplication $*$ satisfies braid relations on $SP^+(G_q)$ (resp. $SP^-(G_q)$); in other words, we have for $\phi_\alpha, \phi_\beta \in SP^+(G_q)$ (resp. $SP^-(G_q)$)*

$$\phi_\alpha * \phi_\beta * \phi_\alpha = \phi_\beta * \phi_\alpha * \phi_\beta, \qquad \langle \alpha, \beta^\upsilon \rangle = \langle \beta, \alpha^\upsilon \rangle = -1;$$

$$\phi_\alpha * \phi_\beta = \phi_\beta * \phi_\alpha, \qquad \langle \alpha, \beta^\upsilon \rangle = \langle \beta, \alpha^\upsilon \rangle = 0.$$

(3) *There is an anti-automorphism i of $BS(G_q)$ such that $i(\phi_x) = \phi_x$ for $x = \alpha, -\alpha \in \Pi(G_q)$.*

Proof. (1) The claims are proved easily. Indeed, the claim that $\phi_\alpha^2 = \phi_\alpha$ (resp. $\phi_{-\alpha}^2 = \phi_{-\alpha}$) follows from the fact that the corresponding algebra

$$k[G_q]/\ker(\phi_\alpha) \qquad \left(\text{resp. } k[G_q]/\ker(\phi_{-\alpha})\right)$$

is even a Hopf algebra. To prove the first claim, let $\{X_{i,j} | i, j \in I\}$ for some index set I be the coordinates of $k[G_q]$, and both

$$\pi_\alpha : k[G_q] \to k[G_q]/\ker(\phi_\alpha) \qquad \text{and}$$

$$\pi_{-\beta} : k[G_q] \to k[G_q]/\ker(\phi_{-\beta})$$

canonical (Hopf algebra) homomorphisms. Put

$$P_0 = \left\{X_{i,j} | i \neq j, \pi_\alpha(X_{i,j}) = 0, \pi_{-\beta}(X_{i,j}) = 0\right\},$$

$$P_1 = \left\{X_{i,i} - 1 | \pi_\alpha(X_{i,i}) = 1, \pi_{-\beta}(X_{i,i}) = 1\right\}.$$

Then both $\ker(\phi_\alpha * \phi_{-\beta})$ and $\ker(\phi_{-\beta} * \phi_\alpha)$ are generated by elements of P_0 and P_1, proving the claim.

(2) The claim for the second case is obvious. To prove the first, we only need to consider such a quantum group G_q that its coordinate algebra $k[G_q]$ is generated by $\{X_{i,j} | 1 \leq i \leq j \leq 3\}$ with the comultiplication

$$\Delta(X_{i,j}) = \sum_{l \geq i,\, j \geq l} X_{i,l} \otimes X_{l,j}$$

for $1 \leq i, j \leq 3$. We can take the α-point ϕ_α in the form of

$$\phi_\alpha(X_{1,1}) = t^{-1}, \qquad \phi_\alpha(X_{1,2}) = x, \qquad \phi_\alpha(X_{2,2}) = t,$$

$$\phi_\alpha(X_{1,3}) = 0, \qquad \phi_\alpha(X_{2,3}) = 0, \qquad \phi_\alpha(X_{3,3}) = 1$$

and the β-point ϕ_β in the form of

$$\phi_\beta(X_{1,1}) = 1, \qquad \phi_\beta(X_{1,2}) = 0, \qquad \phi_\beta(X_{2,2}) = t^{-1},$$

$$\phi_\beta(X_{1,3}) = 0, \qquad \phi_\beta(X_{2,3}) = x, \qquad \phi_\beta(X_{3,3}) = t.$$

Thus, we have

$$\phi_\alpha * \phi_\beta * \phi_\alpha(X_{1,1}) = t^{-1} \otimes 1 \otimes t^{-1},$$

$$\phi_\alpha * \phi_\beta * \phi_\alpha(X_{1,2}) = x \otimes t^{-1} \otimes t + t^{-1} \otimes 1 \otimes x,$$

$$\phi_\alpha * \phi_\beta * \phi_\alpha(X_{2,2}) = t \otimes t^{-1} \otimes t, \qquad \phi_\alpha * \phi_\beta * \phi_\alpha(X_{1,3}) = x \otimes x \otimes 1,$$

$$\phi_\alpha * \phi_\beta * \phi_\alpha(X_{2,3}) = t \otimes x \otimes 1, \qquad \phi_\alpha * \phi_\beta * \phi_\alpha(X_{3,3}) = 1 \otimes t \otimes 1$$

and

$$\phi_\beta * \phi_\alpha * \phi_\beta(X_{1,1}) = 1 \otimes t^{-1} \otimes 1, \qquad \phi_\beta * \phi_\alpha * \phi_\beta(X_{1,2}) = 1 \otimes x \otimes t^{-1},$$

$$\phi_\beta * \phi_\alpha * \phi_\beta(X_{2,2}) = t^{-1} \otimes t \otimes t^{-1}, \qquad \phi_\beta * \phi_\alpha * \phi_\beta(X_{1,3}) = 1 \otimes x \otimes x,$$

$$\phi_\beta * \phi_\alpha * \phi_\beta(X_{2,3}) = t^{-1} \otimes t \otimes x + x \otimes 1 \otimes t,$$

$$\phi_\beta * \phi_\alpha * \phi_\beta(X_{3,3}) = t \otimes 1 \otimes t.$$

Check easily that $\ker(\phi_\beta \phi_\alpha \phi_\beta) = \ker(\phi_\alpha \phi_\beta \phi_\alpha)$ is generated by

$$X_{s,t} \qquad (s < t),$$

$$X_{s,t}X_{s,m} - q^{-1}X_{s,m}X_{s,t} \qquad (t < m),$$

$$X_{t,s}X_{m,s} - q^{-1}X_{m,s}X_{t,s} \qquad (t < m),$$

$$X_{s,t}X_{l,m} - X_{l,m}X_{s,t} \qquad (s < l, t > m),$$

$$X_{1,1}X_{2,2}X_{3,3} - 1,$$

$$X_{s,t}X_{l,m} - X_{l,m}X_{s,t} - (1 - q^2)X_{s,m}X_{t,l} \qquad (s < l, t < m)$$

for $s, t, l, m \in \{1, 2, 3\}$.

(3) Let γ be the antipode of $k[G_q]$. Then the map i, defined by $i(\phi) = \phi\gamma$ for $\phi \in BS(G_q)$, will work, proving the theorem. ∎

Let Q be a semi-group and $g \in Q$. We call g an invariant element of Q if $ga = g$ and $ag = g$ for $a \in Q$.

THEOREM 2.4. *With the above notations, then the invariant element g is unique. Consequently, if i is an anti-automorphism of Q, then $i(g) = g$.*

Proof. The proof is straightforward. ∎

THEOREM 2.5. *Let $\phi \in BS(G_q)$. Then* $\ker(\phi)$ *is a Hopf ideal iff there is a sub-semi-group SG' of $BS(G_q)$ such that ϕ is the invariant element of SG'.*

Proof. First of all, establish notations as follows. For $i \in Z$, put

$$\phi_i = \begin{cases} \phi_{\alpha_i} \in SP(G_q), & \text{if } i > 0; \\ \phi_{-\alpha_{|i|}} \in SP^-(G_q), & \text{if } i < 0. \end{cases}$$

Assume now that $\phi = \phi_{i_1} * \cdots * \phi_{i_s}$ is a reduced expression for ϕ. Let SG' be a sub-semi-group of $BS(G_q)$ generated by $\{\phi_{i_1}, \dots, \phi_{i_s}\}$. Observe that $\ker(\phi)$ is a Hopf ideal if $\phi = \phi * \phi$ and $i(\phi) = \phi$. This is satisfied if ϕ is the invariant element of the semi-group SG'.

Conversely, it is clear that $\ker(\phi) \subset \ker(\phi_{i_j})$ for $j = 1, 2, \ldots, s$. If $\ker(\phi)$ is a Hopf ideal, then either $\phi_{i_j} \in SP(G'_q)$ or $\phi_{i_j} \in SP^-(G'_q)$ according to whether $i_j > 0$ or < 0 for $j = 1, 2, \ldots, s$, and thus, ϕ is the invariant element of the semi-group SG'. Here G'_q is a quantum group with the coordinate algebra $k[G_q]/\ker(\phi)$, completing the proof. ∎

3. THE INVARIANT ELEMENTS OF BOREL SUB-SEMI-GROUPS

In this section, we consider $BS^+(G_q)$, while all results on it apply to $BS^-(G_q)$.

Let $\Pi(G_q) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a simple laced system of simple roots and W the Weyl group corresponding to it. Put $\phi_{\alpha_i} = \phi_i \in SP(G_q)$ for $i \in \{1, 2, \ldots, n\}$.

THEOREM 3.1. *Let w_0 be the longest word in W and $\Phi = \phi_{i_1} * \phi_{i_2} * \cdots * \phi_{i_m} \in BS^+(G_q)$. If $w_0 = \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_m}$, then Φ is the invariant element of $BS^+(G_q)$.*

Proof. The claim follows from the following facts:

(1) Let $\Phi' = \phi_{j_1} * \phi_{j_2} * \cdots * \phi_{j'_m}$. Then $\Phi = \Phi'$ if $w_0 = \alpha_{j_1}\alpha_{j_2} \cdots \alpha_{j'_m}$, by (2) of Theorem 2.3;

(2) For given $\phi_i \in SP(G_q)$, there is a Φ', related to w_0 in the way as in (1), such that $\Phi' = \Phi'' * \phi_i$ or $\Phi' = \phi_i * \Phi''$ for some $\Phi'' \in BS^+(G_q)$;

(3) It follows that $\phi_i^2 = \phi_i$ by (1) of Theorem 2.3 for $i = 1, 2, \ldots, n$.

The proof is completed. ∎

4. THE INVARIANT ELEMENTS OF FUNDAMENTAL SEMI-GROUPS OF RANK 1

In this section, we consider a sub-semi-group $BS(G_q(1))$ of $BS(G_q)$ which is generated by $\{\phi_\alpha, \phi_{-\alpha}\}$ for $\alpha \in \Pi(G_q)$. $BS(G_q(1))$ is usually called a fundamental semi-group of rank 1.

THEOREM 4.1. (1) $\phi_\alpha * \phi_{-\alpha} = \phi_{-\alpha} * \phi_\alpha$.

(2) *$\phi_\alpha * \phi_{-\alpha}$ is the invariant element of $BS(G_q(1))$.*

Proof. To prove (1), we only need to consider such a quantum group G_q that its coordinate algebra $k[G_q]$ is generated by $\{X_{i,j} | 1 \le j, i \le 2\}$ with the comultiplication,

$$\Delta(X_{i,j}) = \sum_{l \le l \le 2} X_{i,l} \otimes X_{l,j}$$

for $1 \leq i, j \leq 2$. Thus, we can take the $-\alpha$-point $\phi_{-\alpha}$ in the form of

$$\phi_{-\alpha}(X_{1,1}) = t^{-1}, \qquad \phi_{-\alpha}(X_{2,1}) = x,$$
$$\phi_{-\alpha}(X_{1,2}) = 0, \qquad \phi_{-\alpha}(X_{2,2}) = t$$

and the α-point ϕ_{α} in the form of

$$\phi_{\alpha}(X_{1,1}) = t^{-1}, \qquad \phi_{\alpha}(X_{2,1}) = 0,$$
$$\phi_{\alpha}(X_{1,2}) = x, \qquad \phi_{\alpha}(X_{2,2}) = t.$$

We have

$$\phi_{-\alpha}\phi_{\alpha}(X_{1,1}) = t^{-1} \otimes t^{-1}, \qquad \phi_{-\alpha}\phi_{\alpha}(X_{2,1}) = x \otimes t^{-1},$$
$$\phi_{-\alpha}\phi_{\alpha}(X_{1,2}) = t^{-1} \otimes x, \qquad \phi_{\alpha}\phi_{-\alpha}(X_{2,2}) = t \otimes t + x \otimes x$$

and

$$\phi_{\alpha}\phi_{-\alpha}(X_{1,1}) = t^{-1} \otimes t^{-1} + x \otimes x, \qquad \phi_{\alpha}\phi_{-\alpha}(X_{2,1}) = t \otimes x,$$
$$\phi_{\alpha}\phi_{-\alpha}(X_{1,2}) = x \otimes t, \qquad \phi_{\alpha}\phi_{-\alpha}(X_{2,2}) = t \otimes t.$$

Check easily that $\ker(\phi_{-\alpha} * \phi_{\alpha}) = \ker(\phi_{\alpha} * \phi_{-\alpha})$ is spanned by

$$X_{s,t}X_{s,m} - q^{-1}X_{s,m}X_{s,t} \qquad (t < m),$$
$$X_{t,s}X_{m,s} - q^{-1}X_{m,s}X_{t,s} \qquad (t < m),$$
$$X_{s,t}X_{l,m} - X_{l,m}X_{s,t} \qquad (s < l,\, t > m),$$
$$X_{s,t}X_{l,m} - X_{l,m}X_{s,t} - (1 - q^2)X_{s,m}X_{t,l} \qquad (s < l,\, t < m),$$
$$X_{1,1}X_{2,2} - q^{-1}X_{1,2}X_{2,1} - 1$$

for $s, t, l, m \in \{1, 2\}$, proving (1). As for (2), we have $K[G_q]/\ker(\phi_{-\alpha} * \phi_{\alpha}) \simeq k[SL_2(q)]$, which is the coordinate algebra of a special quantum group of rank 1, as in [3], completing the proof. ∎

5. THE INVARIANT ELEMENT OF THE FUNDAMENTAL SEMI-GROUP AND A SEMI-SIMPLE QUANTUM GROUP

Assume that

$$\Pi(G_q) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\},$$
$$SP^{+}(G_q) = \{\phi_{\alpha_1}, \phi_{\alpha_2}, \ldots, \phi_{\alpha_n}\} \qquad \text{and}$$
$$SP^{-}(G_q) = \{\phi_{-\alpha_1}, \phi_{-\alpha_2}, \ldots, \phi_{-\alpha_n}\}.$$

Recall in Section 3 that there are ϕ^+ and ϕ^- which are invariant in $BS^+(G_q)$ and $BS^-(G_q)$, respectively.

THEOREM 5.1. *$\phi^+ * \phi^- = \phi^- * \phi^+$. Consequently, $\phi^+ * \phi^-$ is invariant in $BS(G_q)$.*

Proof. Put $\phi_{-\alpha_i} = \phi_{-i}$ and $\phi_{\alpha_i} = \phi_i$ for $i = 1, 2, \dots, n$. To prove the claim, it is sufficient to prove that $\phi_i * \phi_{-j} = \phi_{-j} * \phi_i$ for $i, j = 1, 2, \dots, n$. The claim follows from (1) of Theorem 2.3 for $i \neq j$, while, for $i = j$, this follows from Theorem 4.1. ∎

COROLLARY 5.2. *$BS(G_q)$ is a finite semi-group, more explicitly, we have*

$$\left|BS(G_q)\right| = \left|BS^+(G_q)\right|\left|BS^-(G_q)\right| = \left|BS^+(G_q)\right|^2.$$

Proof. The claim is a direct consequence of Theorem 5.1. ∎

DEFINITION 5.3. Let G_q be as the above. Then, we call G_q a semi-simple quantum group of rank n if $\phi^+ * \phi^- = 0$ where ϕ^+ and ϕ^- are defined as above.

Note that the above definition for $q = 1$ is reconciled with that in algebraic groups.

COROLLARY 5.4. *With the above notations, let G_q be a semi-simple quantum group. Then we have two subgroups B_q^+ and B_q^-, called the positive and negative Borel subgroups, respectively, of G_q: $k[B_q^+] = k[G_q]/\ker(\phi^+)$ and $k[B_q^-] = k[G_q]/\ker(\phi^-)$. Let both $\pi^+ \colon k[G_q] \to k[B_q^+]$ and $\pi^- \colon k[G_q] \to k[B_q^-]$ be canonical homomorphisms of Hopf algebras. Then the algebra homomorphism $(\pi^- \otimes \pi^+)\Delta \colon k[G_q] \to k[B_q^-] \otimes k[B_q^+]$ is injective, where Δ is the comultiplication of $k[G_q]$.*

Proof. The statement follows from Definition 5.3. ∎

6. THE STRUCTURES OF SEMI-GROUPS

In this section, we give the complete relations among generators of $BS(G_q)$, $BS^+(G_q)$, and $BS^-(G_q)$, respectively. Assume that

$$\Pi(G_q) = \{\alpha_1, \alpha_2, \dots, \alpha_n\},$$

$$SP^+(G_q) = \{\phi_1, \phi_2, \dots, \phi_n\} \qquad \text{and}$$

$$SP^-(G_q) = \{\phi_{-1}, \phi_{-2}, \dots, \phi_{-n}\}.$$

Let FSG^+ be a semi-group with a unity e, generated by $\{z_1, z_2, \dots, z_n\}$ subject to the relations

$$z_i^2 = z_i, \qquad 1 \leq i \leq n;$$
$$z_i z_j z_i = z_j z_i z_j, \qquad \langle \alpha_i, \alpha_j^{\upsilon} \rangle = -1;$$
$$z_i z_j = z_j z_i, \qquad \langle \alpha_i, \alpha_j^{\upsilon} \rangle = 0.$$

THEOREM 6.1. *There is an isomorphism of semi-groups σ^+: $FSG^+ \to BS^+(G_q)$ such that $\sigma^+(z_i) = \phi_i$ for $i = 1, 2, \dots, n$.*

Proof. σ^+ as a surjective homomorphism of semi-groups is clear, see [Theorem 2.3]. To prove that σ^+ is an isomorphism, we need to prove that any relation

$$\phi_{i_1} * \phi_{i_2} * \cdots * \phi_{i_s} = \phi_{j_1} * \phi_{j_2} * \cdots * \phi_{j_t}$$

in $BS^+(G_q)$ implies the relation

$$z_{i_1} * z_{i_2} * \cdots * z_{i_s} = z_{j_1} * z_{j_2} * \cdots * z_{j_t}$$

in FSG^+.

Let $FSG^+_{i,j}$ be a sub-semi-group, generated by $\{z_i, z_j\}$ for $i, j \in \{1, 2, \dots, n\}$, of FSG^+. Then there is a canonically surjective semi-group homomorphism $\pi_{i,j}$: $FSG^+ \to FSG^+_{i,j}$, sending z_m to z_m for $m = i, j$, z_m to e for m other than i and j. An important observation is that FSG^+ is isomorphic to $BS^+(G_q)$ in case of $n = 2$. Thus, if a relation

$$\phi_{i_1} * \phi_{i_2} * \cdots * \phi_{i_s} = \phi_{j_1} * \phi_{j_2} * \cdots * \phi_{j_t}$$

follows in $BS^+(G_q)$, then the relation

$$\pi_{i,j}(z_{i_1} * z_{i_2} * \cdots * z_{i_s}) = \pi_{i,j}(z_{j_1} * z_{j_2} * \cdots * z_{j_t})$$

follows in FSG^+ for $i, j \in \{1, 2, \dots, n\}$. This enables us to conclude that

$$z_{i_1} * z_{i_2} * \cdots * z_{i_s} = z_{j_1} * z_{j_2} * \cdots * z_{j_t},$$

proving the theorem. ∎

COROLLARY 6.2. *There is an isomorphism of the semi-group σ^-: $FSG^+ \to BS^-(G_q)$ such that $\sigma^-(z_i) = \phi_{-i} \in SP^-(G_q)$ for $i = 1, 2, \dots, n$.*

Proof. The proof is entirely similar to the above. ∎

THEOREM 6.3. *Let FBG be a semi-group generated by* $\{z_1, z_2, \dots, z_n\}$ *and* $\{z'_1, z'_2, \dots, z'_n\}$ *subject to the relations*

$$z_i z'_j = z'_j z_i, \quad z_i^2 = z_i, \quad (z'_i)^2 = z'_i, \qquad 1 \le i, j \le n;$$
$$z_i z_j z_i = z_j z_i z_j, \qquad \langle \alpha_i, \alpha_j^v \rangle = -1,$$
$$z_i z_j = z_j z_i, \qquad \langle \alpha_i, \alpha_j^v \rangle = 0;$$
$$z'_i z'_j z'_i = z'_j z'_i z'_j, \qquad \langle \alpha_i, \alpha_j^v \rangle = -1,$$
$$z'_i z'_j = z'_j z'_i, \qquad \langle \alpha_i, \alpha_j^v \rangle = 0.$$

Then there is an isomorphism of semi-groups $\sigma\colon FBG \to BS(G_q)$ *such that* $\sigma(z_i) = \phi_i$ *and* $\sigma(z'_i) = \phi_{-i}$ *for* $1 \le i \le n$.

Proof. The statement follows from Theorems 6.1 and 6.2, together with relations $\phi_i \phi_{-j} = \phi_{-j} \phi_i$ for $1 \le i, j \le n$. ∎

7. THE FROBENIUS MORPHISM

In this section, let q be a Nth primitive root of unity and G_q a semi-simple quantum group of rank n with a maximally diagonal torus T. Also, let G be a semi-simple algebraic group with the same maximally diagonal torus as that of G_q and choose the positive subsystem of the root system of G related to T determined by the upper triangle Borel subgroup. Here, we consider G a subgroup of $GL(V)$ for some k-vector space V.

Let $K^0 = k[t_0, t_0^{-1}, x_0]$ be a polynomial algebra. Then there is a homomorphism of algebras $f\colon K_0 \to K$ (cf. Section 1), sending t_0 to t^N and x_0 to x^N. Moreover, K_0 can be endowed with two of the structures of Hopf algebras, denoted by K_0^1 and K_0^2 the corresponding Hopf algebras, such that both $f\colon K_0^1 \to K_1$ and $f\colon K_0^2 \to K_2$ are homomorphisms of Hopf algebras.

LEMMA 7.1. *There are surjective Hopf algebra homomorphisms,*

$$\phi_\alpha^1\colon k[Tg_\alpha] \to K_0^1$$
$$\left(resp.\ \phi_{-\alpha}^1\colon k[Tg_{-\alpha}] \to K_0^2\right)$$

restrictions of which to $k[T]$ *send the unique* α (*resp.* $-\alpha$) *to* t_0^2 (*resp.* t_0^{-2}) *for* $\alpha \in \Pi(G)$. *Here these* g_α *are subgroups of G associated to roots* α.

Proof. It is well known that $k[Tg_\alpha]$ (resp. $k[Tg_{-\alpha}]$) is generated by elements, say $\{t\}$ and x^α (resp. $x^{-\alpha}$). Moreover, among $\{t\}$, there are t_α and t'_α such that

$$\Delta(x^\alpha) = x^\alpha \otimes t_\alpha + t'_\alpha \otimes x^\alpha$$
$$\left(\text{resp. } \Delta(x^{-\alpha}) = x^{-\alpha} \otimes t'_\alpha + t_\alpha \otimes x^{-\alpha}\right).$$

Thus, we can define homomorphisms of Hopf algebras

$$\phi^1_\alpha \colon k[Tg_\alpha] \to K^1_0$$
$$\left(\text{resp. } \phi^1_{-\alpha} \colon k[Tg_{-\alpha}] \to K^2_0\right)$$

as

$$\phi^1_\alpha(t_\alpha) = t_0, \qquad \phi^1_\alpha(t'_\alpha) = t_0^{-1},$$
$$\phi^1_\alpha(t) = 1, \qquad \phi^1_\alpha(x_\alpha) = x_0$$

(resp.

$$\phi^1_{-\alpha}(t_\alpha) = t_0, \qquad \phi^1_{-\alpha}(t'_\alpha) = t_0^{-1},$$
$$\phi^1_{-\alpha}(t) = 1, \qquad \phi^1_{-\alpha}(x_{-\alpha}) = x_0)$$

for $t \in \{t\}$ other than t_α and t'_α, proving the lemma. ∎

THEOREM 7.2. *With the above setup and the assumption that* $\Pi(G_q) = \Pi(G)$, *then there is a homomorphism* F: $k[G] \to k[G_q]$ *of Hopf algebras such that*

(1) *the following commutative diagram follows*:

$$\begin{array}{ccc} k[G] & \xrightarrow[F]{} & k[G_q] \\ \downarrow & & \downarrow \\ k[T] & \xrightarrow{F|_T} & k[T], \end{array}$$

where vertical maps are canonical and $F|_T(t) = t^N$ *for coordinates* $\{t\}$ *of* T.

(2) F *is injective and its images are central.*

Proof. Let G' be a quantum matrix space the coordinate of which is a freely generated k-bialgebra $k[G']$ over coordinates $\{X'_{i,j} | i, j \in J\}$ corresponding to coordinates $\{X_{i,j} | i, j \in J\}$ of $k[G_q]$. Define a homomorphism of algebras F': $k[G'] \to k[G_q]$ sending $X'_{i,j}$ to $X^N_{i,j}$ for $i, j \in J$. Again, let

G be the algebraic group as the above with coordinates $\{x_{i,j} | i, j \in J\}$ in the same index set J and π: $k[G'] \to k[G]$ a homomorphism of bialgebras sending $X'_{i,j}$ to $x_{i,j}$ for $i, j \in J$.

As in Section 2, $F(G')/\Gamma$ is the set of equivalence classes of algebra homomorphisms from $k[G']$: $f \Gamma g$ iff $\ker(f) = \ker(g)$ for $f, g \in k[G']$. Recall that $BS(G_q) = \langle \phi_\alpha, \phi_{-\alpha} | \alpha \in \Pi(G_q) \rangle$ and $BS(G) = \langle \phi_\alpha^1, \phi_{-\alpha}^1 | \alpha \in \Pi(G) \rangle$. Put $BS(G') = \langle \phi_\alpha^1 \pi', \phi_{-\alpha}^1 \pi' | \alpha \in \Pi(G) \rangle \subset F(G')/\Gamma$. Then, there is a map

$$H: \{\phi_\alpha, \phi_{-\alpha} | \alpha \in \Pi(G)\} \to \{\phi_\alpha^1 \pi', \phi_{-\alpha}^1 \pi' | \alpha \in \Pi(G)\}$$

such that $H(\phi_\alpha) = \phi_\alpha^1 \pi'$ and, $H(\phi_{-\alpha}) = \phi_{-\alpha}^1 \pi'$ for $\alpha \in \Pi(G_q) = \Pi(G)$. By Theorem 6.3, H can be extended to a surjective homomorphism of semi-groups from $BS(G_q)$ to $BS(G')$, denoted by the same symbol H. Note that $H(\phi) = \phi F'$ for $\phi \in BS(G_q)$.

Let ϕ^+ and ϕ^- be the invariant elements of sub-semi-groups $BS^+(G_q)$ and $BS^-(G_q)$, respectively. $\phi^+ * \phi^-$ and hence $H(\phi^+) * H(\phi^-)$ are invariant elements of $BS(G_q)$ and $BS(G')$, respectively. Thus, there is a homomorphism of Hopf algebras

$$\begin{aligned} F: k[G] &\simeq k[G']/\ker(H(\phi^+) * H(\phi^-)) \\ &\to k[G_q]/\ker(\phi^+ * \phi^-) \simeq k[G_q], \end{aligned}$$

where the first and third isomorphisms are induced by π and the canonical map, respectively, the second map is induced by F'. Clearly, the following commutative diagram results,

$$\begin{array}{ccc} k[G] & \xrightarrow[F]{} & k[G_q] \\ \downarrow & & \downarrow \\ k[T] & \xrightarrow{F|_T} & k[T] \end{array}$$

where vertical maps are canonical and $F|_T(t) = t^N$ for coordinates $\{t\}$ of T. The injectivity of F follows now from that of $F|_T$ by the virtue of the related result of algebraic groups, see, e.g., Anderson [1].

To complete the proof, that images of F in $k[G_q]$ are central remains to be checked. As images of the map,

$$K[G']/\ker(\phi_x F') = k[G']/\ker(H(\phi_x)) \to k[G_q]/\ker(\phi_x),$$

induced by F', are central for $x = \alpha, -\alpha \in \Pi(G_q) = \Pi(G)$, it is sufficient to prove the following for our claim: if images of both maps

$$\theta_1: k[G']/\ker(H(\phi)) \to k[G_q]/\ker(\phi) \qquad \text{and}$$

$$\theta_2: k[G']/\ker(H(\phi_i)) \to k[G_q]/\ker(\phi_i)$$

induced by F', are central for $\alpha_i \in \Pi(G_q)$ and $\phi \in BS(G_q)$, then images of the map

$$\theta_1 \otimes \theta_2 : k[G']/\ker(H(\phi)) \otimes k[G']/\ker(H(\phi_i)) \\ \to k\left[G_q\right]/\ker(\phi) \otimes k\left[G_q\right]/\ker(\phi_i)$$

are central.

But, this is trivial, proving the theorem. ∎

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